Stochastic Closure in Turbulence

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Outline

Outline

The Deterministic Navier-Stokes Equations

- A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$
\begin{aligned}
u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p \\
u(x,0) &= u_0(x)
\end{aligned}
$$

with the incompressibility condition

$$
\nabla \cdot u = 0,
$$

- Eliminating the pressure using the incompressibility condition gives

$$
\begin{aligned}
u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 \\
u(x,0) &= u_0(x)
\end{aligned}
$$

- The turbulence is quantified by the dimensionless Taylor-Reynolds number $Re_\lambda = \frac{U\lambda}{\nu}$

The Reynolds Decomposition

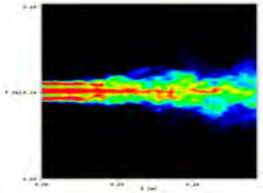
- The velocity is written as $U + u$, pressure as $P + p$
  U describes the large scale flow, u describes the small scale turbulence
- This is the classical Reynolds decomposition (RANS)

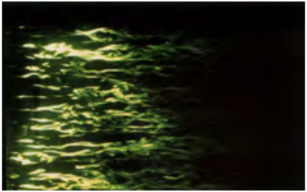
$$U_t + U \cdot \nabla U = \nu \Delta U - \nabla P - \frac{\partial}{\partial x_j} \mathcal{R}_{ij}$$

- The last term the eddy viscosity, where $\mathcal{R}_{ij} = \overline{u_i u_j}$ is the Reynolds stress, describes how the small scale influence the large ones. *Closure problem*: compute $\mathcal{R}_{ij}$.

The Difference between Laminar and Turbulent

- If the Reynolds number is small, only the laminar solution exists
  In this case the ambient noise is quelled
- If the Reynolds number is large, the laminar solution exists but is unstable
  The ambient noise is magnified by the instabilities of the laminar flow and becomes large
- Then the turbulent solution satisfies a stochastic partial differential equation (SPDE)





A Stochastic Closure Theory (SCT)

- Large scale flow

$$
\begin{aligned}
U_t + U \cdot \nabla U &= \nu \Delta U - \nabla P - \frac{\partial}{\partial x_j} \mathcal{R}_{ij} \\
U(x,0) &= U_o(x).
\end{aligned}
$$

- Small scale flow

$$
\begin{aligned}
u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 + \mathrm{Noise} \\
u(x,0) &= u_0(x).
\end{aligned}
$$

- **What is the form of the Noise?** It will contain both additive noise and multiplicative $u \cdot$ noise.

Stochastic Navier-Stokes with Turbulent Noise

- Adding the two types of additive noise and the multiplicative noise we get the stochastic Navier-Stokes equations describing fully developed turbulence

$$
\begin{aligned}
du = (\nu\Delta u &- (U+u)\cdot\nabla u - u\cdot\nabla U + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt \\
&+ \sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0} d_k|k|^{1/3}dt\ e_k(x) \\
&+ u(\sum_{k\neq 0}^{M}\int_{\mathbb{R}} h_k \bar{N}^k(dt,dz)) \qquad (1)\\
u(x,0) &= u_0(x)
\end{aligned}
$$

- Each Fourier component $e_k = e^{2\pi i k\cdot x}$ comes with its own Brownian motion b_t^k and deterministic bound $|k|^{1/3}dt$

The Kolmogorov-Obukhov Theory

- In 1941 Kolmogorov and Obukhov [11, 10, 17] proposed a statistical theory of turbulence
- The structure functions of the velocity differences of a turbulent fluid, should scale with the distance (lag variable) l between them, to the power $p/3$

$$E(|u(x,t) - u(x+l,t)|^p) = S_p = C_p l^{p/3}$$

A. Kolmogorov

A. Obukhov

The Kolmogorov-Obukhov Refinded Similarity with She-Leveque Intermittency Corrections

- The Kolmogorov-Obukhov '41 theory was criticized by Landau for including universal constants C_p and later for not including the influence of the intermittency
- In 1962 Kolmogorov and Obukhov [12, 18] proposed a refined similarity hypothesis

$$S_p = C'_p < \tilde{\epsilon}^{p/3} > l^{p/3} = C_p l^{\zeta_p} \tag{2}$$

l is the lag and ϵ a mean energy dissipation rate

- The scaling exponents

$$\zeta_p = \frac{p}{3} + \tau_p$$

include the She-Leveque intermittency corrections [21] $\tau_p = -\frac{2p}{9} + 2(1 - (2/3)^{p/3})$ and the C_p are not universal but depend on the large flow structure

Why do we need a Statistical Theory of Turbulence?

- Kolmogorov's point of view was that the fluid velocity in turbulence was not a deterministic function but rather a stochastic process
- The reason for this was, that one had to solve the Navier-Stokes equation in a noisy environment to obtain the velocity. This noise had been created by the fluid instabilities magnifying ambient noise. Once the noise was present it could not be ignored
- The consequence is that the only deterministic quantities associated with the turbulent velocity are statistical quantities such as the mean, the variance, the skewness, the kurtosis and so on. We must use probability theory or statistics to study turbulence

The Statistical Theories of Turbulence

One can split turbulence problems into three categories

- Homogeneous Turbulence
- Boundary Value Turbulence (including pipe flow)
- Lagrangian Turbulence

The first two categories are described by an Eulerian observer: the observer is still and the fluid flows past him or her. The third category is described by a Lagrangian observer: she or he follow the fluid particles.

- We will discuss the ST of Homogeneous and Boundary Value Turbulence
- The ST of Lagrangian Turbulence is currently an active field of research

Outline

KOSL Scaling of the Structure Functions, higher order $Re_\lambda \sim 16,000$

Comparison of Theory and Experiments

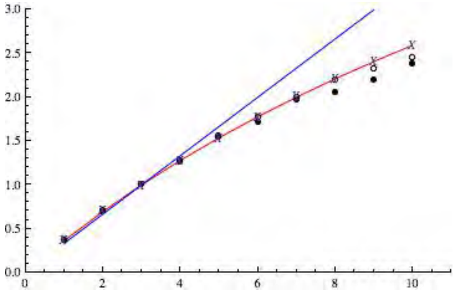


Figure: The exponents of the structure functions as a function of order, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [4], and experiments (X), from [21]. The Kolomogorov-Obukhov '41 scaling is also shown as a blue line for comparion.

Can we compute the Reynolds number dependence of the structure functions?

John Kaminsky in his Ph.D. thesis

$$S_1(x,y,t) = \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|d_k|(1-e^{-\lambda_k t})}{|k|^{\zeta_1} + \frac{4\pi^2\nu}{C}|k|^{\zeta_1+\frac{4}{3}}} |\sin(\pi k \cdot (x-y))|.$$

$$S_2(x,y,t) = \frac{4}{C^2} \sum_{k \in \mathbb{Z}^3} \Big\{ \frac{\frac{C}{2} c_k (1-e^{-2\lambda_k t})}{|k|^{\zeta_2} + \frac{4\pi^2\nu}{C}|k|^{\zeta_2+\frac{4}{3}}} + \frac{d_k^2(1-e^{-\lambda_k t})}{|k|^{\zeta_2} + \frac{8\pi^2\nu}{C}|k|^{\zeta_2+\frac{4}{3}} + \frac{16\pi^4\nu^2}{C^2}|k|^{\zeta_2+\frac{8}{3}}} \Big\} |\sin^2(\pi k \cdot (x-y))|,$$

where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$.

The Variable Density Turbulent Tunnel (VDTT)

- The data comes from the Max Planck Institute for Dynamical and Self-Organization, in Göttingen, Germany (E. Bodenschatz).
- The test sections are about 8 meters long so the turbulence evolves through at least one eddy turnover time, around 1 second.
- This means that the turbulence can be observed over the time that it takes the energy to cascade all the way from the large eddies to the dissipative scale, see [3].
- Measurements were taken from Taylor Reynolds Numbers 110, 264, 508, 1000, and 1450.

Fitting the data

- We have to fit the system size and bring the largest measurements into the range of the structure functions, r/η, where η is the Kolmogorov dissipative scale.
- The largest eddies may be influenced by the system size and need to be modeled.
- The large eddies should scale $c_k \sim b^{-1}$ and $d_k \sim a^{-1}$ for k small.
- The small eddies should scale with k, $c_k \sim k^{-m}$ and $d_k \sim k^{-m}$, for k large.

Results of fits

Taylor Reynolds Number	a	b	D
110	11.6425	0.0161237	1.56917
264	9.58075	0.0523598	1.76897
508	8.31406	0.0650384	1.51799
1000	3.79242	0.0924666	1.32014
1450	2.68367	0.409223	1.3

Table: The fitted values for *a*, *b*, and *D* and *C* below.

Taylor Reynolds Number	110	264	508	1000	1450
Second	2.79532	3.31462	4.20662	7.61993	21.0531
Third	1.40022	1.92759	1.48768	2.7192	3.58878
Fourth	1.0749	1.01212	1.1907	2.35552	5.99954
Sixth	1.15286	1.28604	1.34263	1.73144	2.48915
Eighth	0.615824	.5316486	.596233	1.16513	2.84003

Structure functions for Taylor-Reynolds number 110

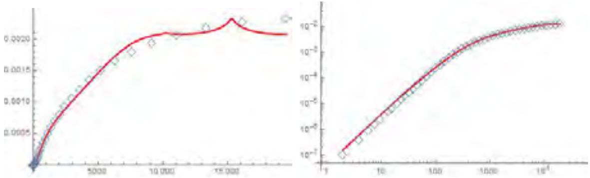


Figure: Second Structure Function, Normal Scale and log-log scale, T-R 110

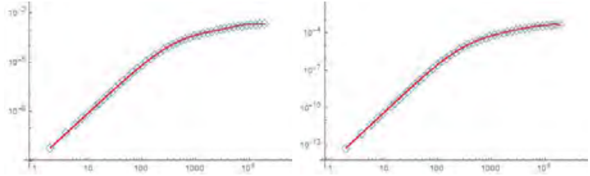


Figure: Third and Fourth Structure function, log-log scale, T-R 110

Structure functions for Taylor-Reynolds number 110 and 1450

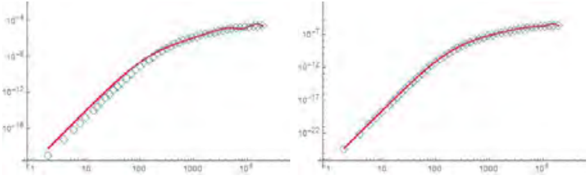


Figure: Sixth and Eighth Structure Function, T-R 110

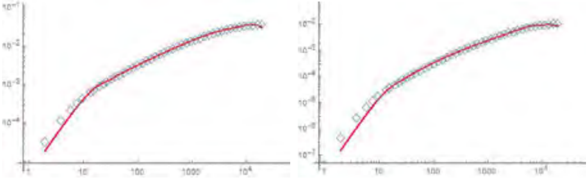


Figure: The Second and Third Structure Function, T-R 1450

Outline

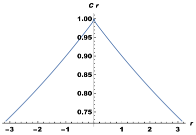
The Noise in Turbulence

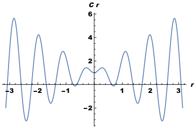
1. The color of the noise depends on the Reynolds number through a, b and the exponent m.
2. For small Reynolds number:

$$C_r = \frac{C}{2} e^{-2\pi br} + \frac{1}{2} e^{-2\pi ar}(r + \frac{1}{2\pi a}),$$

3. For large Reynolds number:

$$C_r = \frac{C}{2} b\cos(2\pi b^2 r) + 2\pi a^2 r \sin(2\pi a^2 r),$$





The Improved SCT Model

The small scale flow satisfies the stochastic Navier-Stokes equation,

$$du + u \cdot \nabla u dt = (\nu \Delta u + \nabla(\Delta^{-1}[\mathrm{Trace}(\nabla u)]))dt - u \cdot \nabla U$$
$$- U \cdot \nabla u + \sum_{k \neq 0} \left(\frac{\mathbf{a}}{|a|^2 + |k|^m} \right) |k|^{-\frac{5}{3}} dt e_k(x)$$
$$+ \sum_{k \neq 0} \frac{\mathbf{b}^{1/2}}{(|b|^2 + |k|^m)^{1/2}} |k|^{-2} db_t^k e_k(x) - u \frac{1}{3} \sum_{k \neq 0} \bar{N}_t^k dt,$$

where $\mathbf{a}, \mathbf{b}^{1/2}, k \in \mathbb{R}^3$, $a = |\mathbf{a}|$, and $b = |\mathbf{b}|$. The improved SCT model depends on three parameters a, b and m, which are all function of the Taylor-Reynolds number Re_λ.

Outline

The Invariant Measure and the Probability Density Functions (PDF)

- Hopf [7] wrote a functional differential equation for the characteristic function of the invariant measure
- The Kolmogorov-Hopf equation for (1) is

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{tr}[P_t C P_t^* \Delta \phi] + \mathrm{tr}[P_t \bar{D} \nabla \phi] + < K(z)P_t, \nabla \phi > \quad (3)$$

where $\bar{D} = (|k|^{1/3} D_k)$, $\phi(z)$ is a bounded function of z,

$$P_t = e^{-\int_0^t \nabla u \; dr} M_t \prod_k^m |k|^{2/3} (2/3)^{N_t^k}$$

- Variance and drift

$$Q_t = \int_0^t e^{K(s)} P_s C P_s^* e^{K^*(s)} ds, \; E_t = \int_0^t e^{K(s)} P_s \bar{D} ds \quad (4)$$

The invariant measure of the stochastic Navier-Stokes

The solution of the Kolmogorov-Hopf equation (3) is

$$R_t\phi(z) = \int_H \phi(e^{Kt}P_t z + EI + y)\mathcal{N}_{(0,Q_t)} * \mathbb{P}_{N_t}(dy)$$

Theorem

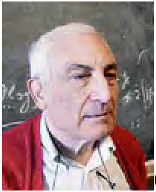
The invariant measure of the Navier-Stokes equation on $H_c = H^{3/2^+}(\mathbb{T}^3)$ is, $\mu(dx) =$

$$e^{<Q^{-1/2}EI,\ Q^{-1/2}x>-\frac{1}{2}|Q^{-1/2}EI|^2}\mathcal{N}_{(0,Q)}(dx)\sum_k \delta_{k,l}\sum_{j=0}^{\infty} p^j_{m_l}\delta_{(N_l-j)}$$

where $Q = Q_\infty$, $E = E_\infty$.

The N-Point Probablity Density of Turbulence

- Finding the invariant measure solves the turbulence problem
- All the (deterministic) statistical properties of the turbulent velocity are determined by the invariant measure
- In particular, the n-point probability density of turbulence is determined by the invariant measure


G. Da Prato


J. Zabczyk

The Probability Density Function (PDF)

Lemma

The PDF is a Normalized Inverse Gaussian distribution NIG of Barndorff-Nilsen [1]:

$$f(x_j) = \frac{(\delta/\gamma)}{\sqrt{2\pi}K_1(\delta\gamma)} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x_j-\mu)^2}\right) e^{\beta(x-\mu)}}{\left(\sqrt{\delta^2 + (x_j-\mu)^2}/\alpha\right)} \qquad (5)$$

where K_1 is modified Bessel's function of the second kind, $\gamma = \sqrt{\alpha^2 - \beta^2}$.

$$f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{\Gamma(1)2e^{\beta\mu}}{(\delta^2+(x-\mu)^2)}, \quad x << 1$$

$$f(x) \sim \frac{(\delta/\gamma)}{2\pi K_1(\delta\gamma)} \frac{e^{\beta(x-\mu)}e^{-\alpha x}}{x^{3/2}}, \quad x >> 1$$

O. Barndorff-Nilsen

The log of the PDF from simulations and fits for the longitudinal direction

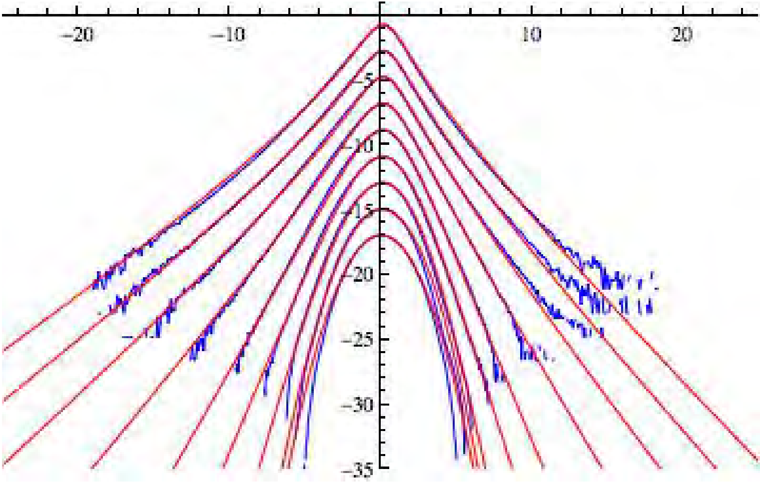


Figure: The log of the PDF from simulations and fits for the longitudinal direction, compare Fig. 4.5 in [23].

Outline

The Three Giants of Boundary Turbulence

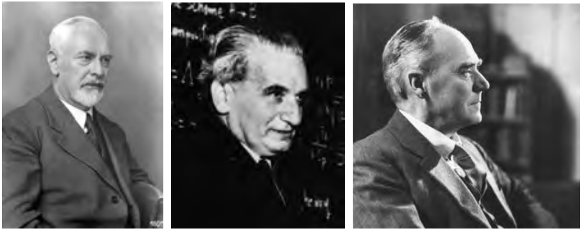


Figure: Ludwig Prandtl, Theodore von Kármán and Geoffrey Taylor. These gentlemen discovered that the turbulent boundary layer could be divided into 4 sublayers, depending on the distance from the boundary, namely

1. The viscous layer (closest to the boundary)
2. The buffer layer (a transition layer, boundary to scaling)
3. The inertial layer (where scaling laws apply)
4. The wake (furthest from the boundary)

Prandtl-von Kármán log-law for fluctuations

- The Prandtl-von Kármán log-law in the inertial range:

$$\langle u \rangle / u_\tau = \kappa^{-1} \ln(y u_\tau / \nu) + B, \tag{6}$$

- $u_\tau = \sqrt{\tau_w / \rho}$ is friction velocity, based on wall stress τ_w, κ the von Kármán constant and B a constant

Figure: The theory was stuck until this gentleman, A. Alan Townsend came along and proposed, in 1956, his hypothesis of "attached eddies".

A Log-Law for the Variation and its Moments

- Townsend [22] proposed that there was a hierarchy of eddies attached to the boundary that transferred momentum into the buffer and inertial layer.
- Townsend hypothesis can be also be used to derive a log law for the streamwise fluctuations $u' = (u - \langle u \rangle)/u_\tau$

$$\langle (u')^2 \rangle^{1/2} = B_2 - A_2 \ln(y/\delta) = D_2(Re_\tau) - A_2 \ln(y^+) \quad (7)$$

- This was verified experimentally by Perry [19, 20] and the coefficients, D_2 and A_2 are called the Townsend-Perry constants.
- Marusic et al. [14, 13, 16, 15], Hultmark et al. [8] proposed a universal log-law, for u':

$$\langle (u')^{2p} \rangle^{1/p} = B_p - A_p \ln(y/\delta) = D_p(Re_\tau) - A_p \ln(y^+) \quad (8)$$

Can we use SCT to compute the Generalized Townsend-Perry Constants?

- These numerical and experimental results on the log dependence of the mean-velocity and the powers of the variation, indicate that the Kolmogorov-Obukhov scaling is dominant in the inertial layer
- Marusic and Meneveau [15], discovered that the Generalized Townsend-Perry were sub-Gaussian
- Can we do the same thing as was possible for homogeneous turbulence and compute all the Townsend-Perry and generalized Townsend-Perry constants?

The coefficients of the log-law for the fluctuations

- The idea of B. and Chen [2] is that in the inertial layer the fluid flow is roughly homogeneous in the streamwise direction
- They make the natural hypothesis that

$$\langle (u')^2 \rangle = \frac{\tau_*}{\kappa u_\tau^2} \ln(y/\delta), \tag{9}$$

where τ_* is the streamwise shear stress
- Then they can evaluate the generalized Townsend-Perry constants and finally they relate τ_* to the structure functions in homogeneous turbulence. This gives
-

$$A_p/A_2 = (l^*)^{\frac{\zeta_p}{p} - \frac{\zeta_2}{2}} (C_p^{1/p}/C_2^{1/2}) \tag{10}$$

where $\zeta_p = p/3 + \tau_p = p/9 + 2(1 - (2/3)^{p/3})$ are the (KOSL) scaling exponents and the C_ps the coefficients of the structure functions, l being the lag variable

Plot of A_p/A_2 as a function of p

We see that the Generalized Townsend-Perry Constant are sub-Gaussian because of the KOSL scaling

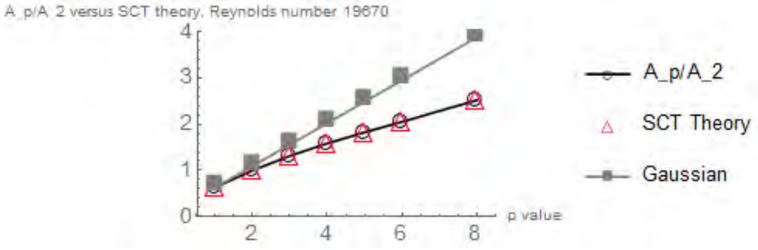


Figure: The first few coefficients A_p, scaled by A_2, as functions of p (red triangles), compared with data (open circles) with Reynolds number $Re_\tau = 19,030$. The gray line and squares represents the Gaussian case. The data was generated in the FPF (Fluid Physics Facility) at the University of New Hampshire, and the red triangles where computed using the SCT [9].

Outline

The Spectral Link for the Generalized Townsend-Perry Constants

- Now that the SCT has given us the fine structure in the inertial layer the question is: can we find the mean flow and the powers of the fluctuation in the other layers?
- Let us start with the answer, as functions of y

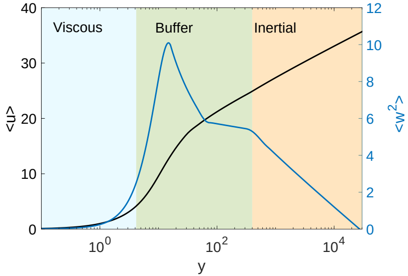


Figure: The mean flow $\langle u \rangle$, black line, and the variation $\langle w^2 \rangle$, blue

The Spectral Function

- Luiza Angheluta showed us work that had been done be a group of physicists at the University of Illinois in Urbana-Champlain, G. Gioia, N. Guttenberg, N. Goldenfeld, and P. Chakraborty [6, 5]
- They had found an expression for the mean flow using a spectral function, that we discovered was the mathematical formulation of Townsend's "attached eddy" hypothesis.
- The spectral function I is the amplitude of the "attached eddy"

$$I\left(\frac{\eta}{s}, \frac{s}{R}\right) = \frac{2}{3}\int_1^\infty e^{-\xi\beta_d\eta/s}\xi^{-5/3}\left(1+\left(\frac{\beta_e s}{R\xi}\right)^2\right)^{-17/6} d\xi. \tag{11}$$

The General Form of the Mean Velocity and Variation

To get the mean velocity and variation across the layer, we must solve the differential equations

$$U' = -\frac{1}{2\kappa^2 l^{3/4} y^2} + \frac{1}{\kappa l^{3/8} y}\sqrt{1 - \frac{y}{Re\sqrt{f}} + \frac{1}{4\kappa^2 l^{3/4} y^2}} \quad (12)$$

with the boundary condition $U = 4.17$ at the beginning of the buffer layer $y = 4.17$. For the fluctuation we have to solve the differential equation for the fluctuation w,

$$w' = \frac{\sqrt{\tau_0} - \sqrt{\langle \tau_0 \rangle}}{\kappa l^{3/8} y \sqrt{\langle \tau_0 \rangle}} \sqrt{1 - \frac{y}{Re\sqrt{f}}}, \quad (13)$$

with the initial condition $w = \frac{\tau_0 - \langle \tau_0 \rangle}{\langle \tau_0 \rangle}\left(4.17 - \frac{17.39}{2Re\sqrt{f}}\right)$, at the beginning of the buffer layer.

The Role of Detached Eddies

- We discover that we do not get the correct form of the variation in the buffer layer with attached eddies only
- We have to include detached eddies also that have a different scaling $1/k$, from the Kolmogorov scaling $k^{-2/3}$
- This scaling corresponds to eddies that are shrinking and speeding up but their energy remains constant
- The first figure shows the comparison of the variation for different Reynolds numbers, experimental and simulation data, with SCT
- The second figure shows the same for the mean velocity with and without the detached eddies.

Comparison of SCT, for the Variation, with Simulations and Experiments

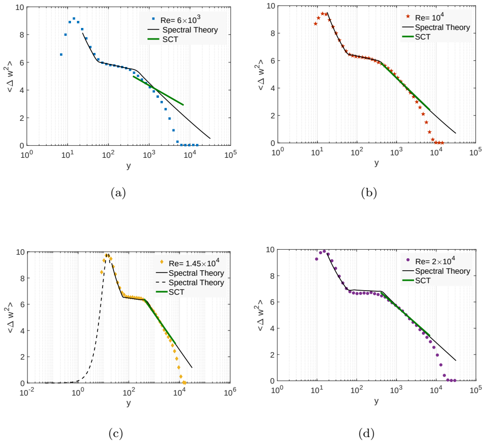


Figure: The variation $\langle w^2 \rangle$ compared with SCT, for different Reynolds numbers

Comparison of SCT, for the Mean Velocity, with Simulations and Experiments

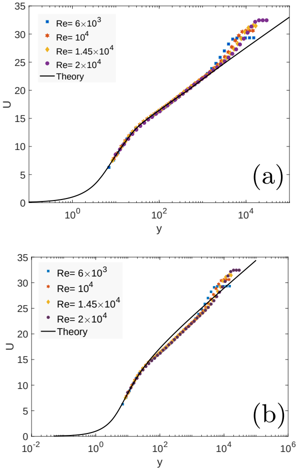


Figure: The mean velocity $\langle u \rangle$ compared with SCT, for different Reynolds numbers, top with, bottom without, detached eddies

Outline

Conclusions

- The stochastic closure theory for Navier-Stokes reproduces the statistical theory of K-O with the intermittency corrections of She-Leveque.
- We computed the dependence of the structure functions of homogeneous turbulence on the Taylor-Reynolds number.
- The theory also produces the n-point probability density and the Generalized Hyperbolic distributions that are the velocity distributions of turbulence.
- SCT extends to boundary flows, and permit a computations of the coefficients in the generalized Prandtl-von Kármán law for the velocity fluctuation
- It also permits the development of the full functional form of the mean velocity, the variations and powers of the variation, across the viscous, buffer and inertial layers, and the wake

The Artist by the Water's Edge

Leonardo da Vinci Observing Turbulence

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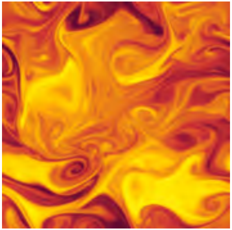
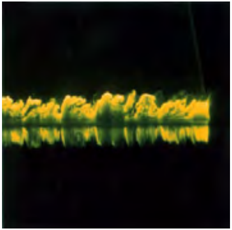
Laminar and Turbulent Flow

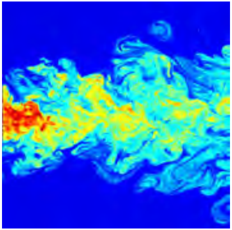
The Reynolds Number : $Re = \frac{UL}{\nu}$

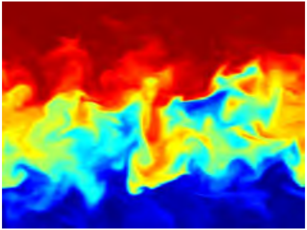
- In 1883 the mechanical engineer Osborne Reynolds observed:
  "The internal motion of water assumes one or other of two broadly distinguishable forms-either the elements of the fluid follow one another along lines of motion which lead in the most direct manner to their destination or they eddy about in sinuous paths the most indirect possible."
- These are respectively laminar and turbulent flow

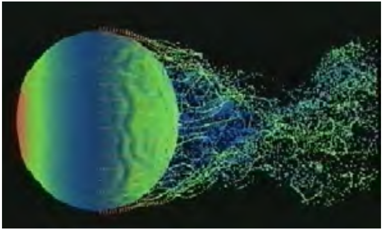
Images of Turbulence



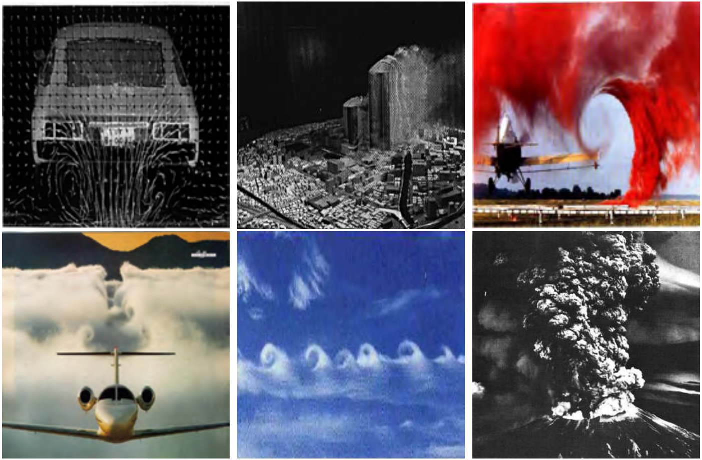








Turbulence in Applications



KOSL Scaling of the Structure Functions, low order $Re_\lambda \sim 16,000$

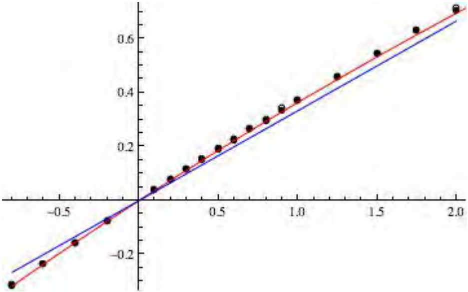


Figure: The exponents of the structure functions as a function of order $(-1, 2]$, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [4]. The Kolmogorov-Obukov '41 scaling is also shown as a blue line for comparion.

Higher order structure functions

The third and pth structure functions are:

$$S_3(x,y,t) = \frac{8}{C^3}\sum_{k\in\mathbb{Z}^3}\Big[\Big\{\frac{\frac{C}{2}c_k|d_k|(1-e^{-2\lambda_k t})(1-e^{-\lambda_k t})}{|k|^{\zeta_3}+\frac{8\pi^2\nu}{C}|k|^{\zeta_3+\frac{4}{3}}+\frac{16\pi^4\nu^2}{C^2}|k|^{\zeta_3+\frac{8}{3}}} + \frac{|d_k|^3(1-e^{-\lambda_k t})^3}{|k|^{\zeta_3}+\frac{12\pi^2\nu}{C}|k|^{\zeta_3+\frac{4}{3}}+\frac{48\pi^4\nu^2}{C^2}|k|^{\zeta_3+\frac{8}{3}}+\frac{64\pi^6\nu^3}{C^3}|k|^{\zeta_3+4}}\Big\}\Big] \times|\sin^3(\pi k\cdot(x-y))|.$$

$$S_p(x,y,t) = \frac{2^p}{C^p}\sum_{k\neq 0} A_p \times |\sin^p[\pi k\cdot(x-y)]|,$$

$$A_p = \frac{2^{\frac{p}{2}}\Gamma(\frac{p+1}{2})\sigma_k^p\,{}_1F_1(-\frac{1}{2}p,\frac{1}{2},-\frac{1}{2}(\frac{M_k}{\sigma_k})^2)}{|k|^{\zeta_p}+\frac{p_k\pi^2\nu}{C}|k|^{\zeta_p+\frac{4}{3}}+\mathcal{O}(\nu^2)},$$

and $M_k = |d_k|(1-e^{-\lambda_k t})$, and $\sigma_k = \sqrt{(\frac{C}{2}c_k(1-e^{-2\lambda_k t}))}$.

Stochastic Closure in Turbulence
Birnir
The Deterministic versus the Stochastic Equation
Homogeneous Turbulence
The Improved Closure Model
The Invariant Measure of Turbulence
Boundary Value Problems
The Spectral Link
Conclusions

- The data comes from the Max Planck Institute for Dynamical and Self-Organization, in Göttingen, Germany (E. Bodenschatz). It was generated by the variable density turbulence tunnel (VDTT).
- The pressurized gases circulate in the VDTT in an upright, closed loop. At the upstream end of two test sections, the free stream is disturbed mechanically.
- The data in the current paper is generated by a fixed grid, but the gas stream can also be disturbed by an active grid resulting in even higher Reynolds number turbulence.

Structure functions for Taylor-Reynolds number 1450

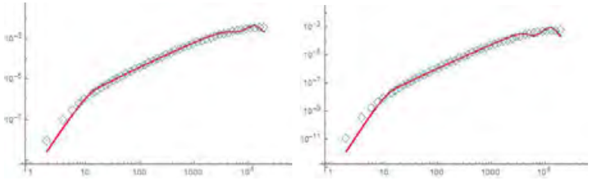


Figure: The Fourth and Sixth Structure Function, log-log scale, T-R 1450

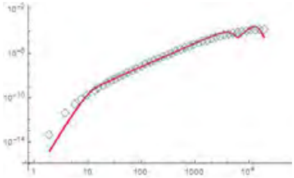


Figure: The Eight Structure function, log-log scale, T-R 1450

Onsager's Observation

The structure functions blow up as $R \to \infty$

- The velocity u, lies in Sobolev space H^s, where $s = \frac{11}{6}$ when intermittency is not taken into account and $s = \frac{29}{18}$ when it is.
- This, in turn, implies that ∇u lies in Sobolev space H^s, where $s = \frac{5}{6}$ without intermittency and $s = \frac{11}{18}$ with intermittency, now $H^s \subset L^p$.
- This follows, by the Sobolev inequality, provided that

$$|\nabla u|_p \leq C \|\nabla u\|_s,$$

- or

$$\frac{5}{6} \geq \frac{3}{2} - \frac{3}{p}.$$

- This is true for $p = 2$, $p = 3$, and $p = 4$, but does not hold for $p = 6$ and $p = 8$.

The Divergence of the Sixth and Eight Structure Functions

The data tell us how rough the fluid velocity is

Taylor Reynolds Number	110	264	508	1000	1450
Second	2.09081	1.49402	1.31448	1.07963	.984291
Third	1.79012	1.41339	1.05553	.822192	.730565
Fourth	1.6408	1.09179	.920749	.687336	.595942
Sixth	1.65727	1.08667	.91658	.681818	.592901
Eighth	1.66164	1.06728	.901549	.662111	.577724

Table: The fitted values for m, uncorrected structure functions.

- The low value of m is due to the divergence of the sine series for the Sixth and the Eight Structure Functions.
- The Fourth structure function sine series diverges with intermittency present.

Formulas for the Mean Velocity and Variation, in the Inertial Layer

- R Reynolds number, η Kolmogorov scale. The spectral function links all attached eddies of radius less than or equal to s. Thus we see how energy is transferred from eddy to eddy into the fluid
- In inertial layer, we get formulas for the averaged velocity U and the variation $\langle w^2 \rangle$, because there $I = 1$
- $$U = \frac{1}{2\kappa^2 y} + \frac{2}{\kappa}\sqrt{1 - \frac{y}{2Re\sqrt{f}}} - \frac{2}{\kappa}\tanh^{-1}\left(\sqrt{1 - \frac{y}{2Re\sqrt{f}}}\right) + K', \quad (14)$$

  $$w = 2\frac{(\sqrt{\tau_0} - \sqrt{\langle \tau_0 \rangle})}{\kappa\sqrt{\langle \tau_0 \rangle}}\sqrt{1 - \frac{y}{2Re\sqrt{f}}} - 2\frac{(\sqrt{\tau_0} - \sqrt{\langle \tau_0 \rangle})}{\kappa\sqrt{\langle \tau_0 \rangle}}\tanh^{-1}\left(\sqrt{1 - \frac{y}{2Re\sqrt{f}}}\right) + C, \quad (15)$$
- These formulas allow us to compute the log law for U and all the even powers of w, and the T-P Constants

Conclusions

- The classical Reynolds decomposition (RANS) of turbulent flow can be closed by a stochastic Navier-Stokes equation for the small scale flow
- The estimate of the structure functions gives the Kolmogorov-Obukhov-She-Leveque scaling, and the intermittency corrections, for homogeneous turbulence
- The eddy viscosity can be computed from the small scale flow, solving the closure problem. This is called the Stochastic Closure Theory (SCT).
- SCT extends to boundary flows, and permit a computations of the coefficients in the generalized Prandtl-von Kármán law for the velocity fluctuation
- It also permits the development of the full functional form of the mean velocity, the variations and powers of the variation, across the viscous, buffer and inertial layers, and the wake